THE VERTEX OF A PARABOLA

If a parabola opens up, then the ***vertex*** of the parabola is the point at the bottom, the minimum point of the parabola. If the parabola opens down, then the vertex is the top, the maximum point of the parabola.

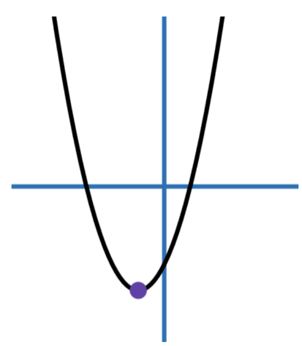
To find the vertex of a parabola in this chapter, we'll need the Midpoint Formula:

If (a, b) and (c, d) are two points in the plane, then the ***midpoint*** *of the line segment connecting those points is*

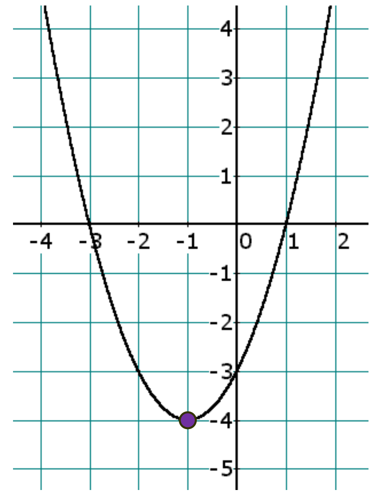
$$\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$$

❑ *FINDING THE VERTEX*

The goal is to devise a way to determine the x-coordinate of the **vertex**, the point at the bottom of the graph of the parabola $y = x^2 + 2x - 3$. Once we find the x-coordinate, we can find the y-coordinate by plugging the x-coordinate into the parabola equation.



Here's the secret to finding the x-coordinate of the vertex: Go straight up from the vertex to the x-axis. That x-value is <u>midway</u> between the x-values of the two x-intercepts; in this case it's midway between $x = -3$ and $x = 1$. Recalling the Midpoint Formula reviewed in the Introduction, the x-coordinate of the vertex is



$$x = \frac{-3+1}{2} = \frac{-2}{2} = -\mathbf{1},$$

a fact which should be clear by looking at the graph. Since the parabola's equation is $y = x^2 + 2x - 3$, we find the y-coordinate of the vertex by simply placing $x = -1$ into the equation and finding y:

$$y = x^2 + 2x - 3 = (-\mathbf{1})^2 + 2(-\mathbf{1}) - 3$$
$$= 1 - 2 - 3 = -4,$$

which is consistent with the picture. We conclude that the vertex of the parabola is the point **(–1, –4)**.

> To find the x-coordinate of the vertex of a parabola, find the <u>average</u> of the x-coordinates of the parabola's x-intercepts.

❑ *EXAMPLES AND FURTHER ANALYSIS*

<u>EXAMPLE 1:</u> **Find the vertex of $y = -2x^2 - 4x + 30$.**

<u>Solution:</u> According to the rule in the box above, we first calculate the x-coordinates of the x-intercepts of the parabola. This, of course, is accomplished by setting y to 0:

$$\mathbf{0} = -2x^2 - 4x + 30$$
$$\Rightarrow \quad 2x^2 + 4x - 30 = 0 \qquad \text{(move everything over)}$$
$$\Rightarrow \quad 2(x^2 + 2x - 15) = 0 \qquad \text{(factor out the GCF)}$$
$$\Rightarrow \quad 2(x + 5)(x - 3) = 0 \qquad \text{(factor the trinomial)}$$
$$\Rightarrow \quad (x + 5)(x - 3) = 0 \qquad \text{(divide both sides by 2)}$$

$\Rightarrow \quad x + 5 = 0 \text{ or } x - 3 = 0$ (set each factor to 0)

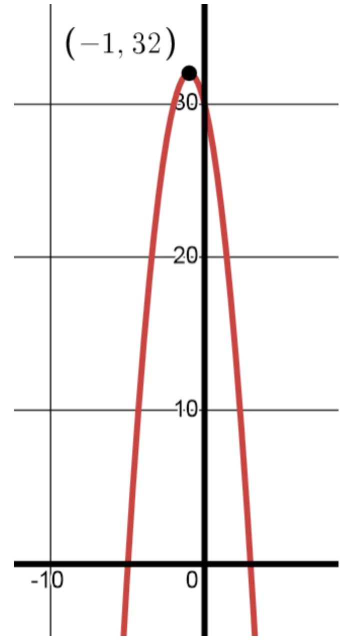
$\Rightarrow \quad x = -5 \text{ or } x = 3$ (solve each linear equation)

The average of these x-values is $\frac{-5+3}{2} = \frac{-2}{2} = -1$, and so we have $x = -1$ as the x-coordinate of the vertex. The y-value of the vertex is

$$y = -2x^2 - 4x + 30 = -2(\mathbf{-1})^2 - 4(\mathbf{-1}) + 30$$
$$= -2 + 4 + 30 = 32$$

and we conclude that the vertex of the parabola is the point

$$\boxed{(-1, 32)}$$



EXAMPLE 2: **Find the vertex of the parabola $y = x^2 - 6x + 9$.**

Solution: Find the x-intercepts by setting y to 0:

$\mathbf{0} = x^2 - 6x + 9$

$\Rightarrow \quad 0 = (x - 3)(x - 3)$

$\Rightarrow \quad x - 3 = 0 \text{ or } x - 3 = 0$

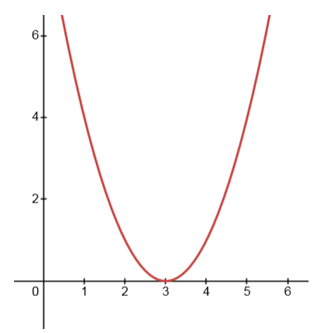
$\Rightarrow \quad x = 3 \text{ or } x = 3$

There's only one x-intercept, (3, 0), but it did occur twice! So we'll go through the motions and calculate the average of 3 and 3, which is **3**, the x-coordinate of the vertex of the parabola. And the y-coordinate of the vertex is

$$y = \mathbf{3}^2 - 6(\mathbf{3}) + 9 = 9 - 18 + 9 = \mathbf{0}$$

We conclude that the vertex of the parabola is

$$\boxed{(3, 0)}$$



<u>EXAMPLE 3:</u> **Find the vertex of $y = 3x^2 - 7x + 3$.**

<u>Solution:</u> Set y to 0 to calculate the x-intercepts. This will be followed by averaging those x-intercepts to find the x-coordinate of the vertex of the parabola.

$$\mathbf{0} = 3x^2 - 7x + 3$$

Factoring will prove fruitless, so it's time for the Quadratic Formula (we could also Complete the Square) — the values of a, b, and c are 3, –7, and 3:

$$x = \frac{-(\mathbf{-7}) \pm \sqrt{(\mathbf{-7})^2 - 4(\mathbf{3})(\mathbf{3})}}{2(\mathbf{3})} = \frac{7 \pm \sqrt{49-36}}{6} = \frac{7 \pm \sqrt{13}}{6}$$

Thus, the x-coordinates of the x-intercepts are

$$\frac{7+\sqrt{13}}{6} \quad \text{and} \quad \frac{7-\sqrt{13}}{6}$$

It is these two numbers that we average to obtain the x-coordinate of the vertex:

$$x = \frac{\frac{7+\sqrt{13}}{6} + \frac{7-\sqrt{13}}{6}}{2} \qquad \text{(average = sum divided by 2)}$$

$$= \frac{\frac{7+\sqrt{13}+7-\sqrt{13}}{6}}{2} \qquad \text{(adding the two fractions)}$$

$$= \frac{\frac{14}{6}}{2} = \frac{\frac{7}{3}}{2} = \frac{7}{3} \div 2 = \frac{7}{3} \times \frac{1}{2} = \frac{7}{6}$$

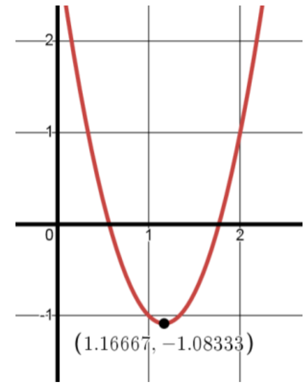
Now we know that the x-coordinate of the vertex is $\frac{7}{6}$, and so the y-coordinate of the vertex is

$$y = 3\left(\frac{\mathbf{7}}{\mathbf{6}}\right)^2 - 7\left(\frac{\mathbf{7}}{\mathbf{6}}\right) + 3 = 3\left(\frac{49}{36}\right) - 7\left(\frac{7}{6}\right) + 3 = \frac{49}{12} - \frac{49}{6} + 3$$

$$= \frac{49}{12} - \frac{98}{12} + \frac{36}{12} = -\frac{13}{12}$$

We're finally able to state our conclusion:
The vertex of the parabola is

$$\boxed{\left(\frac{7}{6}, -\frac{13}{12}\right)}$$



EXAMPLE 4: **Find the vertex of $y = x^2 + 3x + 4$.**

Solution: Seems innocent enough — what could possibly go wrong? (Ever heard of Murphy's Law?) To find x-intercepts, we set y to 0 and obtain the quadratic equation

$$\mathbf{0} = x^2 + 3x + 4$$

Since this quadratic is not factorable, we employ the Quadratic Formula:

$$x^2 + 3x + 4 = 0 \qquad a = 1;\ b = 3;\ c = 4$$

$$\Rightarrow \quad x = \frac{-\mathbf{3} \pm \sqrt{(\mathbf{3})^2 - 4(\mathbf{1})(\mathbf{4})}}{2(\mathbf{1})} = \frac{-3 \pm \sqrt{9-16}}{2} = \frac{-3 \pm \sqrt{-7}}{2}$$

Gee, these two numbers are not real numbers; you might know that we call them "imaginary" numbers, and that they don't exist anywhere on the x-axis (which is an axis of real numbers). We deduce that <u>the parabola has no x-intercepts</u>. So how in the heck do we average the x-intercepts to find the vertex when they don't even exist? The answer is, trust me! That is, let's calculate the average of these two "phantom" intercepts and see what happens.

There are two issues to consider if we want to calculate the average of two imaginary numbers. First, the two solutions of the quadratic equation above can be written separately as

$$\frac{-3 + \sqrt{-7}}{2} \quad \text{and} \quad \frac{-3 - \sqrt{-7}}{2}$$

Second, the average of two numbers (even these imaginary numbers!) is found by dividing their sum by 2 — here goes:

$$x = \frac{\frac{-3+\sqrt{-7}}{2} + \frac{-3-\sqrt{-7}}{2}}{2} \qquad \text{(average = sum divided by 2)}$$

$$= \frac{\frac{-3+\sqrt{-7}-3-\sqrt{-7}}{2}}{2} \qquad \text{(adding the two fractions)}$$

$$= \frac{\frac{-6}{2}}{2} \qquad \text{(combine like terms)}$$

$$= \frac{-3}{2} = -\mathbf{\frac{3}{2}}$$

Thus, assuming you still trust me, we will accept this number to be the x-coordinate of the vertex. The y-coordinate would be

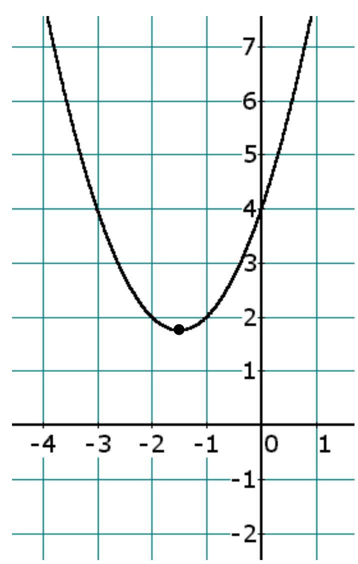
$$y = x^2 + 3x + 4 = \left(-\mathbf{\frac{3}{2}}\right)^2 + 3\left(-\mathbf{\frac{3}{2}}\right) + 4 = \frac{9}{4} - \frac{9}{2} + 4$$

$$= \frac{9}{4} - \frac{18}{4} + \frac{16}{4} = \mathbf{\frac{7}{4}}$$

Our best theory, therefore, is that the vertex of the parabola is

$$\boxed{\left(-\frac{3}{2}, \frac{7}{4}\right)}$$

Let's confirm that this vertex we just calculated of the parabola with no x-intercepts makes some sense by sketching the parabola. It looks pretty reasonable. Notice that the parabola truly has no x-intercepts, just as predicted by the algebra. Moreover, if you look carefully at the vertex of the parabola, you can estimate that the x-coordinate is between -2 and -1, which the number $-\frac{3}{2}$ is. Also, the



y-coordinate of the vertex appears to be a little shy of 2, which the number $\frac{7}{4}$ is. I'm convinced — are you?

Homework

1. By averaging the x-coordinates of the x-intercepts, find the **vertex** of each parabola:

   a. $y = x^2 + 2x - 48$ b. $y = x^2 + 10x + 25$

   c. $y = 2x^2 + 8x + 5$ d. $y = 3x^2 - 6x + 4$

❑ *PUTTING IT ALL TOGETHER*

EXAMPLE 5: **Graph $y = x^2 - 6x + 5$ using all the concepts of we've learned about parabolas.**

Solution: The leading coefficient, 1, is positive; thus, the parabola **opens up**.

To find x-intercepts, set y to 0:

$$x^2 - 6x + 5 = \mathbf{0}$$

$$\Rightarrow \quad (x-5)(x-1) = 0$$

$$\Rightarrow \quad x - 5 = 0 \text{ or } x - 1 = 0$$

$$\Rightarrow \quad x = 5 \text{ or } x = 1$$

Therefore, **the x-intercepts are (1, 0) and (5, 0)**.

As for y-intercepts, we set x to 0:

$$y = \mathbf{0}^2 - 6(\mathbf{0}) + 5 = 5$$

Thus, **the y-intercept is (0, 5)**.

To find the x-coordinate of the vertex we calculate the average of the two x-coordinates of the two x-intercepts:

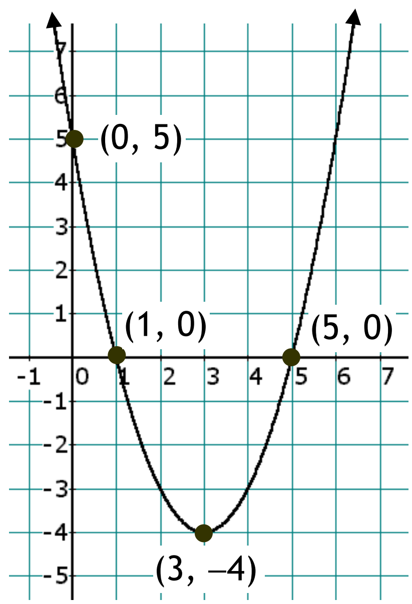
$$x = \frac{1+5}{2} = \frac{6}{2} = 3,$$

which implies that

$$y = \mathbf{3}^2 - 6(\mathbf{3}) + 5 = 9 - 18 + 5 = -4$$

So **the vertex of the parabola is (3, –4)**.

We now have our parabola:



Homework

2. Find the y-intercept, the x-intercept(s), and the vertex of $y = 2x^2 + 7x - 5$. Sketch the parabola.

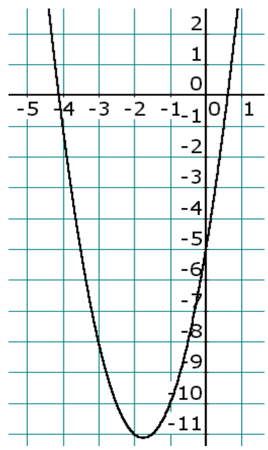
3. Find the y-intercept, the x-intercept(s), and the vertex of $y = -16x^2 + 24x - 9$. Sketch the parabola.

4. Find the y-intercept, the x-intercept(s), and the vertex of $y = x^2 - 3x + 5$. Sketch the parabola.

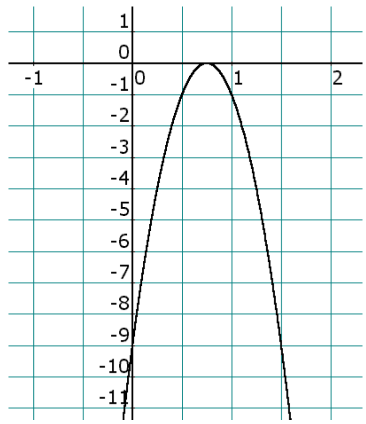
Solutions

1. a. $(-1, -49)$ b. $(-5, 0)$ c. $(-2, -3)$ d. $(1, 1)$

2. y-int: $(0, -5)$ x-int: $\left(\frac{-7+\sqrt{89}}{4}, 0\right)$, $\left(\frac{-7-\sqrt{89}}{4}, 0\right)$ $V\left(-\frac{7}{4}, -\frac{89}{8}\right)$

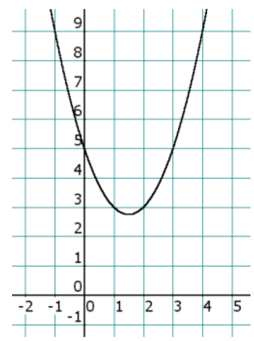


3. y-int: $(0, -9)$ x-int: $\left(\frac{3}{4}, 0\right)$ $V\left(\frac{3}{4}, 0\right)$



Note that the axes are scaled differently. In actuality, the parabola is much skinnier than the one shown.

4. y-int: $(0, 5)$ x-int: None $V\left(\frac{3}{2}, \frac{11}{4}\right)$



"EDUCATION IS THE KEY TO UNLOCK THE GOLDEN DOOR OF FREEDOM."

– George Washington Carver